Mathematical Excalibur

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Olympiad Corner

The 7th China Hong Kong Math Olympiad took place on December 4, 2004. Here are the problems.

Problem 1. For $n \ge 2$, let $a_1, a_2, \ldots, a_n, a_{n+1}$ be positive and $a_2 - a_1 = a_3 - a_2 = \cdots = a_{n+1} - a_n \ge 0$. Prove that

$$\frac{1}{a_2^2} + \frac{1}{a_3^2} + \cdots + \frac{1}{a_n^2} \le \frac{n-1}{2} \cdot \frac{a_1 a_n + a_2 a_{n+1}}{a_1 a_2 a_n a_{n+1}}.$$

Determine when equality holds.

Problem 2. In a school there are b teachers and c students. Suppose that
(i) each teacher teaches exactly k students; and
(ii) for each pair of distinct students, exactly h teachers teach both of them.

Show that $\dfrac{b}{h} = \dfrac{c(c-1)}{k(k-1)}$.

Problem 3. On the sides AB and AC of triangle ABC, there are points P and Q respectively such that $\angle APC = \angle AQB = 45°$. Let the perpendicular line to side AB through P intersects line BQ at S. Let the perpendicular line to side AC through Q intersects line CP at R. Let D be on side BC such that $AD \perp BC$.

(continued on page 4)

Acknowledgment: Thanks to Elina Chiu, Math. Dept., HKUST for general assistance.

On-line:
http://www.math.ust.hk/mathematical_excalibur/

The editors welcome contributions from all teachers and students. With your submission, please include your name, address, school, email, telephone and fax numbers (if available). Electronic submissions, especially in MS Word, are encouraged. The deadline for receiving material for the next issue is ***March 31, 2005***.

For individual subscription for the next five issues for the 03-04 academic year, send us five stamped self-addressed envelopes. Send all correspondence to:

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例析數學競賽中的計數問題 (一)

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組合數學中的計數問題，數學競賽題中的熟面孔，看似司空見慣，不足爲奇．很多同學認爲只要憑藉單純的課內知識就可左右逢源，迎刃而解．其實具體解題時，卻會使你挖空心思，也無所適從．對於這類問題往往首先要通過構造法描繪出對象的簡單數學模型，繼而借助在計數問題中常用的一些數學原理方可得出所求對象的總數或其範圍．

1 運用分類計數原理與分步計數原理

分類計數原理與分步計數原理（即加法原理與乘法原理）是關於計數的兩個基本原理，是解決競賽中計數問題的基礎．下面提出的三個問題，注意結合排列與組合的相關知識，構造出相應的模型再去分析求解．

<u>例 1</u> 已知兩個實數集合 $A=\{a_1,a_2,\cdots,a_{100}\}$ 與 $B=\{b_1,b_2,\cdots,b_{50}\}$，若從$A$到$B$的映射$f$使得$B$中每個元素都有原象，且 $f(a_1) \le f(a_2) \le \ldots \le f(a_{100})$，則這樣的映射共有（ ）個．

（A）C_{100}^{50}（B）C_{99}^{48}（C）C_{100}^{49}（D）C_{99}^{49}

<u>解答</u> 設 $b_1,b_2,\cdots,b_{50}$ 按從小到大排列爲 $c_1<c_2<\cdots<c_{50}$（因集合元素具有互異性，故其中不含相等情形）．

將A中元素 $a_1,a_2,\cdots,a_{100}$ 分成 50 組，每組依次與B中元素 $c_1,c_2,\cdots,c_{50}$ 對應．這裏，我們用 $a_1a_2a_3c_1a_4a_5c_2\cdots$，表示 $f(a_1)=f(a_2)=f(a_3)=c_1$，$f(a_4)=f(a_5)=c_2$，⋯

考慮 $f(a_1) \le f(a_2) \le \cdots \le f(a_{100})$，容易得到 $f(a_{100})=c_{50}$，這就是說 c_{50} 只能寫在 a_{100} 的右邊，故只需在 $a_1□\ a_2□\ a_3□\ \cdots□\ a_{98}□\ a_{99}□\ a_{100}c_{50}$ 之間的 99 個空位 "□" 中選擇 49 個位置並按從左到右的順序依次填上 $c_1,c_2,\cdots,c_{49}$．從而構成滿足題設要求的映射共有 C_{99}^{49} 個．因此選 D．

<u>例 2</u> 有人要上樓，此人每步能向上走 1 階或 2 階，如果一層樓有 18 階，他上一層樓有多少種不同的走法？

<u>解答 1</u> 此人上樓最多走 18 步，最少走 9 步．這裏用 $a_{18},a_{17},a_{16},\cdots,a_9$ 分別表示此人上樓走 18 步，17 步，16 步，⋯，9 步時走法（對於任意前後兩者的步數，因後者少走 2 步 1 階，而多走 1 步 2 階，計後者少走 1 步）的計數結果．考慮步子中的每步 2 階情形，易得 $a_{18}=C_{18}^0$，$a_{17}=C_{17}^1$，$a_{16}=C_{16}^2$，⋯，$a_9=C_9^9$．

綜上，他上一層樓共有 $C_{18}^0+C_{17}^1+C_{16}^2+\cdots+C_9^9=1+17+120+\cdots+1=4181$ 種不同的走法．

<u>解答 2</u> 設 F_n 表示上n階的走法的計數結果．

顯然，$F_1=1$，$F_2=2$（2 步 1 階；1 步 2 階）．對於 $F_3,F_4,\cdots$，起步只有兩種不同走法：上 1 階或上 2 階．

因此對於 F_3，第 1 步上 1 階的情形，還剩 $3-1=2$ 階，計 F_2 種不同的走法；對於第 1 步上 2 階的情形，還剩 $3-2=1$ 階，計 F_1 種不同的走法．總計 $F_3=F_2+F_1=2+1=3$．

同理，$F_4=F_3+F_2=3+2=5$，$F_5=F_4+F_3=5+3=8$，⋯，$F_{18}=F_{17}+F_{16}=2584+1597=4181$．

<u>例 3</u> 在世界盃足球賽前，F國教練爲了考察 $A_1,A_2,\ldots,A_7$ 這七名隊員，準備讓他們在三場訓練比賽（每場 90 分鐘）都上場．假設在比賽的任何時刻，這

些隊員中有且僅有一人在場上，並且 A_1,A_2,A_3,A_4 每人上場的總時間（以分鐘爲單位）均被 7 整除，A_5,A_6,A_7 每人上場的總時間（以分鐘爲單位）均被 13 整除．如果每場換人次數不限，那麼按每名隊員上場的總時間計算，共有多少種不同的情況．

解答　設 $A_i\left(i=1,2,\cdots,7\right)$ 上場的總時間分別爲 $a_i\left(i=1,2,\cdots,7\right)$ 分鐘．

根據題意，可設 $a_i=7k_i\left(i=1,2,3,4\right)$，$a_i=13k_i\left(i=5,6,7\right)$，其中 $k_i\left(i=1,2,\cdots,7\right)\in Z^+$．

令 $\sum_{i=1}^{4}k_i=m$，$\sum_{i=5}^{7}k_i=n$，其中 $m\geq 4$，$n\geq 3$，且 $m,n\in Z^+$．則 $7m+13n=270$．易得其一個整數特解爲 $\begin{cases}m=33\\n=3\end{cases}$，又因 $(7,13)=1$，故其整數通解爲 $\begin{cases}m=33+13t\\n=3-7t\end{cases}$．再由 $\begin{cases}33+13t\geq 4\\3-7t\geq 3\end{cases}$，得 $-\frac{29}{13}\leq t\leq 0$，故整數 $t=0,-1,-2$．

從而其滿足條件的所有整數解爲 $\begin{cases}m=33,\\n=3;\end{cases}\begin{cases}m=20,\\n=10;\end{cases}\begin{cases}m=7,\\n=17.\end{cases}$

對於 $\sum_{i=1}^{4}k_i=33$ 的正整數解，可以寫一橫排共計 33 個 1，在每相鄰兩個 1 之間共 32 個空位中任選 3 個填入“+”號，再把 3 個“+”號分隔開的 4 個部分裏的 1 分別統計，就可得到其一個正整數解，故 $\sum_{i=1}^{4}k_i=33$ 有 C_{32}^3 個正整數解 $\left(k_1,k_2,k_3,k_4\right)$；同理 $\sum_{i=5}^{7}k_i=3$ 有 C_2^2 個正整數解 $\left(k_5,k_6,k_7\right)$；從而此時滿足條件的正整數解 $\left(k_1,k_2,k_3,k_4,k_5,k_6,k_7\right)$ 有 $C_{32}^3\cdot C_2^2$ 個．…

因此滿足條件的所有正整數解 $\left(k_1,k_2,k_3,k_4,k_5,k_6,k_7\right)$ 有

$C_{32}^3\cdot C_2^2+C_{19}^3\cdot C_9^2+C_6^3\cdot C_{16}^2=42244$

個，即按每名隊員上場的總時間計算，共有 42244 種不同的情況．

2 運用容斥原理

容斥原理，又稱包含排斥原理或逐步淘汰原理．顧名思義，即先計算一個較大集合的元素的個數，再把多計算的那一部分去掉．它由英國數學家 J.J.西爾維斯特首先創立．這個原理有多種表達形式，其中最基本的形式爲：

設 $A_1,A_2,\cdots,A_n$ 是任意 n 個有限集合，以 *card* (*S*) 代表 *S* 的元素的個數，則

$$card\left(A_1\cup A_2\cup\cdots\cup A_n\right)$$
$$=\sum_{1\leq i\leq n}card(A_i)-\sum_{1\leq i<j\leq n}card(A_i\cap A_j)$$
$$+\sum_{1\leq i<j<k\leq n}card\left(A_i\cap A_j\cap A_k\right)-\cdots$$
$$+(-1)^{n-1}card(A_1\cap A_2\cap\cdots\cap A_n).$$

例 4　由數字 1，2，3 組成 n 位數，且在這個 n 位數中，1，2，3 的每一個至少出現一次，問這樣的 n 位數有多少個？

解答　設 U 是由 1，2，3 組成的 n 位元數的集合，A_1 是 U 中不含數字 1 的 n 位元數的集合，A_2 是 U 中不含數字 2 的 n 位元數的集合，A_3 是 U 中不含數字 3 的 n 位元數的集合，則

$card\left(U\right)=3^n$,
$card\left(A_1\right)=card\left(A_2\right)=card\left(A_3\right)=2^n$,
$card\left(A_1\cap A_2\right)=card\left(A_2\cap A_3\right)=card\left(A_3\cap A_1\right)=1$,
$card\left(A_1\cap A_2\cap A_3\right)=0.$ ．

因此

$$card(U)-card(A_1\cup A_2\cup A_3)$$
$$=3^n-3\cdot 2^n+3\cdot 1-0=3^n-3\cdot 2^n+3$$ ．

即符合題意的 n 位數的個數爲 $3^n-3\cdot 2^n+3$ ．

下面，我們再來看一個關於容斥原理應用的變異問題．

例 5　參加大型團體表演的學生共 240 名，他們面對教練站成一行，自左至右按 1，2，3，4，5，…依次報數．教練要求全體學生牢記各自所報的數，並做下列動作：先讓報的數是 3 的倍數的全體同學向後轉；接著讓報的數是 5 的倍數的全體同學向後轉；最後讓報的數是 7 的倍數的全體同學向後轉．問：

⑴此時還有多少名同學面對教練？

⑵面對教練的同學中，自左至右第 66 位同學所報的數是幾？

解答　⑴設 $U=\{1,2,3,\cdots,240\}$，A_i 表示由 U 中所有 i 的倍數組成的集合．則

$$card\left(U\right)=240,\quad card\left(A_3\right)=\left[\frac{240}{3}\right]=80,$$
$$card\left(A_5\right)=\left[\frac{240}{5}\right]=48,\ card\left(A_7\right)=\left[\frac{240}{7}\right]=34$$
$$card\left(A_{15}\right)=\left[\frac{240}{15}\right]=16,\ card\left(A_{21}\right)=\left[\frac{240}{21}\right]=11,$$
$$card\left(A_{35}\right)=\left[\frac{240}{35}\right]=6,\ card\left(A_{105}\right)=\left[\frac{240}{105}\right]=2.$$

從而此時有

$card(U)-[card(A_3)+card(A_5)+card(A_7)]$
$+2[card(A_{15})+card(A_{21})+card(A_{35})]$
$-4card(A_{105})=136$

名同學面對教練．

如果我們借助威恩圖進行分析，利用上面所得數據分別填入圖 1，注意按從內到外的順序填．

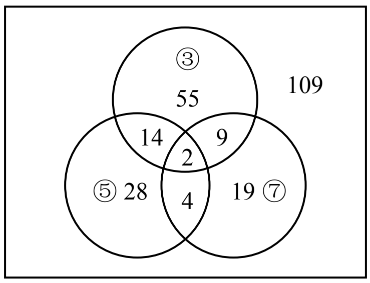

圖 1

如圖 1，此時面對教練的同學一目了然，應有 109+14+4+9=136 名．

(continued on page 4)

Problem Corner

We welcome readers to submit their solutions to the problems posed below for publication consideration. The solutions should be preceded by the solver's name, home (or email) address and school affiliation. Please send submissions to *Dr. Kin Y. Li, Department of Mathematics, The Hong Kong University of Science & Technology, Clear Water Bay, Kowloon, Hong Kong.* The deadline for submitting solutions is ***March 31, 2005.***

Problem 216. (*Due to Alfred Eckstein, Arad, Romania*) Solve the equation

$$4x^6 - 6x^2 + 2\sqrt{2} = 0.$$

Problem 217. Prove that there exist infinitely many positive integers which cannot be represented in the form

$$x_1^3 + x_2^5 + x_3^7 + x_4^9 + x_5^{11},$$

where x_1, x_2, x_3, x_4, x_5 are positive integers. (*Source:* 2002 *Belarussian Mathematical Olympiad, Final Round*)

Problem 218. Let O and P be distinct points on a plane. Let $ABCD$ be a parallelogram on the same plane such that its diagonals intersect at O. Suppose P is not on the reflection of line AB with respect to line CD. Let M and N be the midpoints of segments AP and BP respectively. Let Q be the intersection of lines MC and ND. Prove that P, Q, O are collinear and the point Q does not depend on the choice of parallelogram $ABCD$. (*Source:* 2004 *National Math Olympiad in Slovenia, First Round*)

Problem 219. (*Due to Dorin Mărghidanu, Coleg. Nat. "A.I. Cuza", Corabia, Romania)* The sequences $a_0,a_1,a_2,\ldots$ and $b_0,b_1,b_2,\ldots$ are defined as follows: $a_0,b_0 > 0$ and

$$a_{n+1} = a_n + \frac{1}{2b_n},\quad b_{n+1} = b_n + \frac{1}{2a_n}$$

for $n = 1,2,3,\ldots$. Prove that

$$\max\{a_{2004}, b_{2004}\} > \sqrt{2005}.$$

Problem 220. (*Due to Cheng HAO, The Second High School Attached to Beijing Normal University*) For $i = 1,2,\ldots, n$, and $k \ge 4$, let $A_i = (a_{i1}, a_{i2}, \ldots , a_{ik})$ with $a_{ij} = 0$ or 1 and every A_i has at least 3 of the k coordinates equal 1. Define the distance between A_i and A_j to be

$$\sum_{m=1}^{k} |a_{im} - a_{jm}|.$$

If the distance between any A_i and A_j ($i \ne j$) is greater than 2, then prove that

$$n \le 2^{k-3} - 1.$$

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Solutions

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Problem 211. For every a, b, c, d in [1,2], prove that

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \le 4\,\frac{a+c}{b+d}.$$

(*Source: 32[nd] Ukranian Math Olympiad*)

Solution. **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece) and **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania).

Since $0 < b + d \le 4$, it suffices to show

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \le a + c.$$

Without loss of generality, we may assume $1 \le a \le c$, say $c = a + e$ with $e \ge 0$. Then

$$\frac{a+b}{b+c} + \frac{c+d}{d+a} \le 1 + \left(1 + \frac{e}{d+a}\right)$$

$$\le 2 + e$$

$$\le 2a + e = a + c.$$

In passing, we observe that equality holds if and only if $e = 0$, $a = c = 1$, $b = d = 2$.

Other commended solvers: **CHENG Hei** (Tsuen Wan Government Secondary School, Form 5), **LAW Yau Pui** (Carmel Divine Grace Foundation Secondary School, Form 6) and **YIM Wing Yin** (South Tuen Mun Government Secondary School, Form 5).

Problem 212. Find the largest positive integer N such that if S is any set of 21 points on a circle C, then there exist N arcs of C whose endpoints lie in S and each of the arcs has measure not exceeding 120°.

Solution.

We will $N = 100$. To see that $N \le 100$, consider a diameter AB of C. Place 11 points close to A and 10 points close to B. The number of desired arcs is then

$$\binom{11}{2} + \binom{10}{2} = 100.$$

To see that $N \ge 100$, we need to observe that for every set T of $k = 21$ points on C, there exists a point X in T such that there are at least $[(k-1)/2]$ arcs XY (with Y in T, $Y \ne X$) each having measure not exceeding 120°. This is because we can divide the circle C into three arcs C_1, C_2, C_3 of 120° (only overlapping at endpoints) such that the common endpoint of C_1 and C_2 is a point X of T. If X does not have the required property, then there are $1 + [(k-1)/2]$ points of T lies on C_3 and any of them can serve as X.

Next we remove X and apply the same argument to $k = 20$, then remove that point, and repeat with $k = 19, 18, \ldots, 3$. We get a total of $10 + 9 + 9 + 8 + 8 + \ldots + 1 + 1 = 100$ arcs.

Problem 213. Prove that the set of all positive integers can be partitioned into 100 nonempty subsets such that if three positive integers a, b, c satisfy $a + 99b = c$, then at least two of them belong to the same subset.

Solution. **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Let $f(n)$ be the <u>*largest*</u> nonnegative integer k for which n is divisible by 2^k. Then given three positive integers a, b, c satisfying $a + 99\,b = c$ at least two of $f(a)$, $f(b)$, $f(c)$ are equal. To prove this, if $f(a) = f(b)$, then we are done. If $f(a) < f(b)$, then $f(c) = f(a)$. If $f(a) > f(b)$, then $f(c) = f(b)$.

Therefore, the following partition suffices:

$$S_i = \{n \mid f(n) \equiv i \pmod{100}\}$$

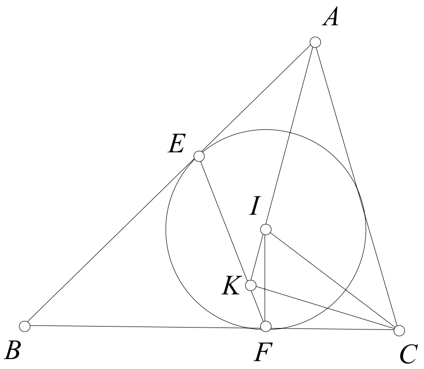
for $1 \le i \le 100$.

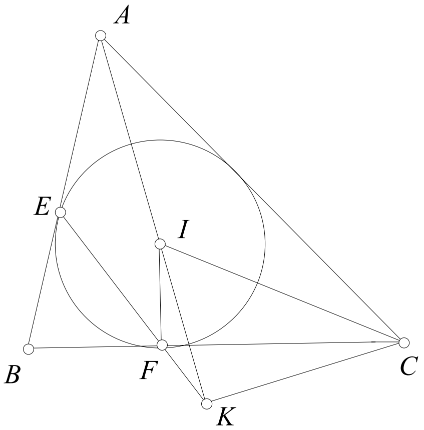
Problem 214. Let the inscribed circle of triangle ABC be tangent to sides AB, BC at E and F respectively. Let the angle bisector of $\angle CAB$ intersect segment EF at K. Prove that $\angle CKA$ is a right angle.

Solution. **CHENG Hei** (Tsuen Wan Government Secondary School, Form 5), **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece), **YIM Wing Yin** (South Tuen Mun

Government Secondary School, Form 5) and **YUNG Ka Chun** (Carmel Divine Grace Foundation Secondary School, Form 6).

Most of the solvers pointed out that the problem is still true if the angle bisector of $\angle CAB$ intersect line EF at K *outside the segment EF*. So we have two figures.





Let I be the center of the inscribed circle. Then A, I, K are collinear. Now $\angle CIK = ½(\angle BAC + \angle ACB)$. Next, $BE = BF$ implies that $\angle BFE = 90° - ½\ \angle CBA = ½\ (\angle BAC + \angle ACB) = \angle CIK$. (In the second figure, we have $\angle CFK = \angle BFE = \angle CIK$.) Hence C, I, K, F are concyclic. Therefore, $\angle CKI = \angle CFI = 90°$.

Other commended solvers: **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1).

Problem 215. Given a 8×8 board. Determine all squares such that if each one is removed, then the remaining 63 squares can be covered by 21 3×1 rectangles.

Solution. **CHEUNG Yun Kuen** (HKUST, Math Major, Year 1).

Let us number the squares of the board from 1 to 64, with 1 to 8 on the first row, 9 to 16 on the second row and so on.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Using this numbering, a 3×1 rectangle will cover three numbers with a sum divisible by 3. Since $64 \equiv 1 \pmod 3$, only squares with numbers congruent to 1 (mod 3) need to be considered for our problem.

If there is a desired square for the problem, then considering the left-right symmetry of the board and the up-down symmetry of the board, the images of a desired square under these symmetries are also desired squares. Hence they must also have numbers congruent to 1 (mod 3) in them.

However, the only such square and its image squares having this property are the squares with numbers 19, 22, 43 and 46.

Finally square 19 has the required property (and hence also squares 22, 43, 46 by symmetry) by putting 3×1 rectangles as shown in the following figure (those squares having the same letter are covered by the same 3×1 rectangle).

A	A	A	B	B	B	F	G
C	C	C	D	D	D	F	G
H	I		E	E	E	F	G
H	I	J	J	J	K	K	K
H	I	L	L	L	M	M	M
N	O	P	Q	R	S	T	U
N	O	P	Q	R	S	T	U
N	O	P	Q	R	S	T	U

Other commended solvers: **HUDREA Mihail** (High School "Tiberiu Popoviciu" Cluj-Napoca Romania), **NG Siu Hong** (Carmel Divine Grace Foundation Secondary School, Form 6) and **Achilleas P. PORFYRIADIS** (American College of Thessaloniki "Anatolia", Thessaloniki, Greece).

Olympiad Corner

(continued from page 1)

Problem 3. *(cont.)* Prove that the lines PS, AD, QR meet at a common point and lines SR and BC are parallel.

Problem 4. Let $S = \{1, 2, \ldots, 100\}$. Determine the number of functions $f: S \to S$ satisfying the following conditions.

(i) $f(1) = 1$;
(ii) f is bijective (i.e. for every y in S, the equation $f(x) = y$ has exactly one solution);
(iii) $f(n) = f(g(n))\,f(h(n))$ for every n in S.

Here $g(n)$ and $h(n)$ denote the uniquely determined positive integers such that $g(n) \le h(n)$, $g(n)\,h(n) = n$ and $h(n) - g(n)$ is as small as possible. (For instance, $g(80) = 8$, $h(80) = 10$ and $g(81) = h(81) = 9$.)

例析數學競賽中的計數問題 (一)

(continued from page 2)

⑵用上面類似的方法可算得自左至右第 1 號至第 105 號同學中面對教練的有 60 名．

考慮所報的數不是 3，5，7 的倍數的同學沒有轉動，他們面對教練；所報的數是 3，5，7 中的兩個數的倍數的同學經過兩次轉動，他們仍面對教練；其餘同學轉動了一次或三次，都背對教練．

作如下分析：106，107，~~108~~（3 的倍數），109，~~110~~（5 的倍數），~~111~~（3 的倍數），~~112~~（7 的倍數），113，~~114~~（3 的倍數），~~115~~（5 的倍數），116，~~117~~（3 的倍數），118，~~119~~（7 的倍數），120（3、5 的倍數），……，可知面對教練的第 66 位同學所報的數是 118.

(*to be continued*)